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On the Heat Transfer in Rayleigh–Bénard Systems

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In this paper we discuss some theoretical aspects concerning the scaling laws of the Nusselt number versus the Rayleigh number in a Rayleigh–Bénard cell. We present a new set of numerical simulations and compare our findings against the predictions of existing models. We then propose a new theory which relies on the hypothesis of Bolgiano scaling. Our approach generalizes the one proposed by Kadanoff, Libchaber, and coworkers and solves some of the inconsistencies raised in the recent literature.

KEY WORDS: Turbulent flows; convection; heat transfer; Rayleigh–Bénard system; natural convection.

I. INTRODUCTION

In this paper we discuss the scaling properties of heat transport in Rayleigh–Bénard systems. Although this topic has received a lot of attention during the last years, still there are controversial interpretations of experimental results. In 1989, Libchaber, Kadanoff and co-workers[(1)] have shown that a new and unpredicted scaling range is observed in Rayleigh–Bénard systems at large enough Rayleigh numbers. They also proposed a physical picture of the experimental observations, based on the crucial role played by "plumes", the coherent structures present in thermal convection, in heat transport. This picture was somehow questioned by Shraiman and Siggia, who proposed a rather different theoretical approach. Both models make relatively ad hoc assumptions on qualitative features of the fluid motion in appropriate sub-domains of a Rayleigh–Bénard cell. These

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assumptions, although reasonable on physical ground, are not explicitly justified by the still poorly understood dynamics, in a system of rather complex phenomenology.

In this paper, we try to build a more dynamical picture of the problem, by establishing a link between the scaling properties of heat transport and other well controlled dynamical features encountered in convective turbulence. We start presenting and discussing a new set of results based on direct numerical simulation which, in our opinion, are able to shed more light on the problem. We then generalize the model of Castaing *et al.*[1] with a simple ansatz, whose basic physical meaning is that the dynamics that control Bolgiano scaling in convective turbulence also modulates the scaling of the Nusselt number. Our ansatz leads to well defined predictions that we have verified numerically (and could be experimentally checked). The obvious objection that Bolgiano scaling (expected to set in at scales larger than the Bolgiano length) can hardly play any role in heat transfer (where scales of the order of the boundary layers are obviously important) can be solved by observing, as already pointed out in ref. 10, that quantities usually regarded as globally characterizing the flow (such as energy or temperature dissipation, Bolgiano length) can still be locally defined in a convective cell as a function of the vertical coordinate. In particular, Bolgiano length decreases sharply in regions close to the boundary layer.

Our paper is organized as follows. In Section II we introduce the equation of motion, the dimensionless parameters describing the problem and Bolgiano scaling. In Section III we review the theoretical models proposed by Castaing *et al.* and Shraiman and Siggia emphasizing the physical assumptions underlying the two different approach. In Section IV we present our numerical simulations and in Section V, the most important part of the present work, we discuss and justify a simple ansatz which generalizes the model proposed in ref. 1. In Section VI we present our conclusions.

II. THE PROBLEM

We consider a fluid in a rectangular cell of horizontal size L and vertical size H. The fluid is heated from below and cooled from above by contact with two heat reservoirs. The temperature field $T(x, y, z; t)$ satisfies the boundary conditions:

$$T|_{z=0} = \bar{T} - \frac{\Delta T}{2} \qquad T|_{z=H} = \bar{T} + \frac{\Delta T}{2} \tag{1}$$

Furthermore the vertical walls are supposed to be adiabatic (i.e., thermal exchange through these walls are supposed to be negligible). We also assume that the fluid flow satisfies the Boussinesq equation of motion (see ref. 2):

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + (\mathbf{v}(\mathbf{x}, t) \cdot \nabla)\, \mathbf{v}(\mathbf{x}, t) = -\frac{1}{\rho} \nabla p + \alpha g \theta \hat{z} + \nu\, \Delta \mathbf{v}(\mathbf{x}, t) \tag{2}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{v}(\mathbf{x}, t) \cdot \nabla)\, \theta = \chi\, \Delta \theta \tag{3}$$

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0 \tag{4}$$

where g is the acceleration of gravity, α the volume thermal expansion coefficient, ν and χ kinematic and thermal diffusivity, $\theta \equiv T - \bar{T}$. The Boussinesq equations of motion are a first order approximation of buoyancy effects and are supposed to be accurate if thermal gradients are not too strong. Furthermore the incompressibility condition is valid if velocities are much smaller than sound speed.

The relevant dimensionless parameters in Eqs. (2) and (3) are the Rayleigh number Ra, the Prandtl number Pr and the aspect ratio Γ:

$$Ra \equiv \frac{\alpha g\, \Delta T\, H^3}{\nu \chi}; \qquad Pr \equiv \frac{\nu}{\chi}; \qquad \Gamma \equiv \frac{H}{L} \tag{5}$$

For large enough values of Ra, the flow becomes turbulent, i.e., chaotic both in time and in space. One of the basic issues that we want to discuss in this paper concerns how much heat is transferred from the bottom boundary to the top boundary. To this end it is useful to introduce the dimensionless number:

$$Nu \equiv \frac{\langle v_z T \rangle - \chi(\partial \langle T \rangle / \partial z)}{(\chi\, \Delta T/H)} \equiv \frac{\langle \omega' \theta' \rangle}{\chi(\Delta T/H)} \tag{6}$$

where ω' is the turbulent fluctuation of the vertical velocity, θ' the (turbulent) temperature fluctuation and $\langle \cdots \rangle$ stands for space and time averages. Nu is called the Nusselt number. It measures the amount of heat transported by turbulent fluctuations with respect to the heat transport due to molecular motion ($\chi\, \Delta T/H$ is the heat flux due to conduction if the fluid were at rest).

We can now state our problem in terms of the dimensionless parameters so far introduced: we want to investigate the functional relationship,

$$Nu = f(Ra, Pr, \Gamma) \tag{7}$$

To our knowledge, no meaningful effect has been found experimentally on turbulent heat transport due to the geometric parameter Γ. Therefore, we shall neglect, hereafter, the dependency on Γ in (7). We shall confine most of our discussion to the $Pr = 1$ case.

If the fluid were at rest the temperature drop inside the cell would be linear. Because of convection the mean temperature profile differs from the purely conductive one. In particular one find that almost all the temperature drop occurs in the thermal boundary layers. The thickness of the thermal boundary layer, λ, can be roughly defined as the thickness of the layer over which the temperature drop is nearly equal to $\Delta T/2$. An important relation (experimentally well verified) allows us to connect λ to the Nu number:

$$Nu \sim \frac{H}{\lambda} \tag{8}$$

We remark that relation (8) becomes rigorously true if one defines λ as follows:

$$\frac{1}{\lambda} = \frac{1}{\Delta T} \frac{\partial \langle T \rangle}{\partial z}\bigg|_{z=0} \tag{9}$$

By means of the relation (8) our problem can be rephrased as the understanding of the scaling of λ versus Ra.

A related issue that we want to address is the role of buoyancy forces effects on the statistical properties of turbulent fluctuations for Rayleigh–Bénard convection. Let us introduce the velocity and temperature difference defined as:

$$\delta v_{//}(r) = [\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \hat{r} \tag{10}$$

$$\delta T(r) = T(x + r) - T(x) \tag{11}$$

Following Yakhot[12] we can write:

$$\langle \delta v_{//}(r)^3 \rangle = -\frac{4}{5} \varepsilon r + \frac{2\alpha g}{r^4} \int_o^r r'^4 \langle \delta T(r') \, \delta v_{//}(r') \rangle \, dr' + 6\nu \frac{\partial}{\partial r} \langle \delta v(r)^2 \rangle \tag{12}$$

$$\langle \delta v_{//}(r) \, \delta T(r)^2 \rangle = -\frac{4}{3} Nr + \frac{2}{r^2} \int_o^r y^2 \langle \delta T(y) \, \delta v_z(y) \rangle \, dy \frac{\partial \theta}{\partial z} + 6\chi \frac{\partial}{\partial r} \langle \delta T(r)^2 \rangle \tag{13}$$

where ε is the mean rate of energy dissipation, $\varepsilon = \nu/2 \int \sum_{i,j} (\partial_i v_j + \partial_j v_i)^2 \, d^3x$, and N is the mean rate of temperature dissipation, $N = \chi/2 \int \sum_i (\partial_i T)^2 \, d^3x$.

Equations (12) and (13) replace the well known "4/5" Kolmogorov equation for homogeneous and isotropic turbulence. These equations have been derived by assuming that small scale turbulent fluctuations in Rayleigh–Bénard system are, to first approximation, homogeneous and isotropic (see Yakhot[12]). The second term on the right hand side of (12) and (13) represents however a non isotropic, thermally driven, contribution. Therefore (12) and (13), although cannot be proven rigorously, can be useful as guidelines for describing the difference, if any, between homogeneous isotropic turbulence and thermal turbulence.

Neglecting intermittency effects, from (13) we have the scaling relation:

$$\delta v(r)\,\delta T(r)^2 \simeq Nr \tag{14}$$

There are two physically interesting limit regimes in Eq. (12): either the first or the second term on the right hand side dominates. In the latter case we deduce the following balance:

$$\delta v^3(r) \sim \alpha g\,\delta T(r)\,\delta v(r)\cdot r \tag{15}$$

By using (14) we obtain:

$$\delta v^2(r) \sim \alpha g r\,\delta T(r) \simeq \alpha g r\left(\frac{Nr}{\delta v(r)}\right)^{1/2} \tag{16}$$

We can equivalently rephrase Eqs. (15)–(16) as scaling laws for the velocity and temperature increments:

$$\delta v(r) \simeq (\alpha g)^{2/5}\,N^{1/5} r^{3/5} \tag{17}$$

$$\delta T(r) \simeq (\alpha g)^{-1/5}\,N^{2/5} r^{1/5} \tag{18}$$

The scaling property defined by (17)–(18) is referred to as Bolgiano scaling[13]. Using Bolgiano scaling, we can evaluate the second term of the right hand side of (12). We obtain:

$$\frac{2\alpha g}{r^4}\int_0^r r'^4\langle\delta T(r')\,\delta v(r')\rangle\,dr' \simeq \alpha g\,\delta T(r)\,\delta v(r)\cdot r \simeq (\alpha g)^{6/5}\,N^{3/5} r^{9/5} \tag{19}$$

From (19) we obtain a consistency condition, requiring that the second term of the r.h.s. in (12) is larger than εr. This is true for scales $r > L_B$, where:

$$L_B \equiv \frac{\varepsilon^{5/4}}{N^{3/4}(\alpha g)^{3/2}} \tag{20}$$

L_B is called the Bolgiano scale. For $r < L_B$, the statistical properties of thermal turbulence should be described by Kolmogorov theory of turbulence with scaling:

$$\delta v(r) \simeq \varepsilon^{1/3} r^{1/3} \tag{21}$$

$$\delta T(r) \simeq N^{1/2} \varepsilon^{-1/6} r^{1/3} \tag{22}$$

We remark that Eq. (22) corresponds to the scaling of a passive scalar while Eq. (21) is the usual Kolmogorov (1941) prediction. Finally let us notice that for $r > L_B$ the first term on the r.h.s. of (13) is always greater than the second term, which represents a consistency condition.

The dependence of both N and ε on Nu and Ra can be exactly derived from the equation of motion. We have (see ref. 5 for details):

$$\langle \varepsilon \rangle = \frac{\nu \chi^2}{H^4} (Nu - 1) \cdot Ra \tag{23}$$

$$\langle N \rangle = Nu \cdot Ra^2 \frac{\chi^3 \nu^2}{H^8 (\alpha g)^2} \tag{24}$$

Beside any rigorous derivations, we can give a physical meaning to (23), (24) using the following arguments. Consider first the mean rate of temperature dissipation. Since almost all the temperature gradient (ΔT) is across the thermal boundary layer (of thickness λ) we can estimate:

$$N \sim \left(\frac{\lambda}{H}\right) \left(\frac{\Delta T}{\lambda}\right)^2 \sim Nu \cdot Ra^2 \tag{25}$$

On the other hand, by using Eq. (23), we obtain:

$$\delta v(L_B) \cdot \delta T(L_B) \simeq Nu \cdot Ra \tag{26}$$

which is equivalent to say that the Bolgiano length L_B can be interpreted as the characteristic scale of the eddies which transport heat in a Rayleigh–Bénard system.

Evidences of Bolgiano scaling (17) and (18) have been reported both in 2D[7] and 3D[8] numerical simulations of Rayleigh–Bénard systems (see ref. 9 for a detailed description including the effects of intermittency).

III. REVIEW OF PROPOSED SCALING THEORIES ON HEAT TRANSPORT

In this section we review some theoretical models, proposed in the past, to derive the scaling properties of Nu vs. Ra (for a review see also ref. 3). Our emphasis is on the physical assumptions underlying the models, rather than on rigorous derivations from the equations of motion.

We first discuss three different arguments leading to the old (and currently experimentally disproved) scaling relation $Nu \sim Ra^{1/3}$. Such relation was predicted by many authors among them Malkus,[19, 20] Priestley,[21] Howard,[22, 23] Spiegel.[24]

The first argument assumes that the boundary layer is marginally stable, i.e., that the effective $Ra(\lambda)$ number computed at the thermal boundary layer thickness, λ, is equal to a critical Rayleigh number Ra_c, independent of Ra. Since by definition, $Ra = \alpha g \, \Delta T \, H^3/\nu\chi$, we get $Ra(\lambda) = R \cdot (\lambda/H)^3$ and hence $Ra = Ra_c(H/\lambda)^3$. Therefore, $\lambda \sim H(Ra_c/Ra)^{1/3}$. Using (8) we finally obtain the result $Nu \sim Ra^{1/3}$.

Another way to reach the same result is the assumption of the independence of the heat flux from the height of the cell, H. Supposing a scaling of the form $Nu \sim Ra^\gamma$, the Nusselt number is just $F/(\chi \, \Delta T/H) \sim (\alpha g \, \Delta T \, H^3/\nu\chi)^\gamma$, γ is required to be 1/3, for the heat flux F be independent of H. From this argument we understand that every model which leads to a 1/3 exponent, does, implicitly, assume a decoupling of the top and bottom boundary layers.

Finally, we want to point out a rather simple argument proposed in ref. 12 leading to the same result. Let us assume that, close to the thermal boundary layer, the statistical properties of turbulent fluctuations are not affected by buoyancy effect. This implies, that we can use the standard dimensional analysis of Kolmogorov theory. Let us also assume that the thermal boundary layer thickness, λ, equals the viscous boundary layer thickness, η, (i.e., $\lambda \sim \eta$). Assuming K41 scaling for velocity differences, $\delta v(r) \sim (\varepsilon r)^{1/3}$, from the condition $Re \sim 1 \sim (\delta v(\eta) \, \eta)/\nu$ and from the scaling $\varepsilon \sim Nu \cdot Ra$ we obtain $Nu \sim Ra^{1/3}$. We remark that this argument is self-consistent, because $\varepsilon \sim Ra^{4/3}$, $N \sim Ra^{7/3}$ so that $L_B \sim Ra^{-1/12}$, which implies $\eta \sim \varepsilon^{-1/4} \sim Ra^{-1/3}$ and $\lambda \sim \eta \leqslant L_B$. The above argument can also be derived by estimating the energy dissipation $\varepsilon(\lambda)$ in the thermal boundary layer by the Kolmogorov relation $\varepsilon \simeq u(\lambda)^3/\lambda$, where $u(\lambda)^2 \simeq (\alpha g \, \Delta T \, \lambda)$. By assuming that most of the energy dissipation takes place in the thermal boundary layer and using (24) we obtain $Nu \sim Ra^{1/3}$.

We observe that in all cases, for scales of order λ and Ra large enough, $\lambda \ll L_B$. Thus Bolgiano scaling should not be applied near by the boundary layer.

In an important paper, Castaing *et al.*[1] reported for the first time clear evidence that

$$Nu \sim Ra^\gamma; \qquad \gamma \simeq 0.281 \simeq \tfrac{2}{7} \tag{27}$$

Castaing *et al.* (1989) have shown that this scaling law is valid for $Ra \geqslant 10^6$. Further experimental results (Ciliberto[5]) have shown that $\gamma \sim 2/7$ even for lower Ra. The results reported in Castaing *et al.* have motivated many experimental and theoretical efforts aimed at understanding the physical mechanism leading to the (unexpected) scaling (27). Here we review two rather different theoretical model proposed in Castaing *et al.*[1] and Shraiman–Siggia.[4]

The Castaing *et al.*[1] theory is based upon the assumption that there exist three layers (hereafter referred to as A, B and C) characterized by different physical properties:

A-layer: the thermal boundary layer, near the bottom and top boundaries, of thickness λ and temperature differences $\Delta T/2$. In the A-layer, instabilities generate plumes, of typical size λ, which are expelled into the B-layer.

B-layer: a mixing region, of thickness much greater that λ and smaller than the size of the cell H. In this layer thermal plumes are accelerated due to buoyancy effect. We shall indicate by δT and δv the characteristic size of temperature and velocity fluctuations, respectively.

C-layer: the central region of the cell of size comparable with the size of the system. Velocity and temperature fluctuations will be indicated by u_c and T_c respectively. In this layer thermal plumes are advected with almost constant velocity.

The physical picture behind this theory is the following. Thermal plumes are generated in the A-layer, accelerated in the B-layer and advected in the C-layer. The Nusselt number can be estimated at the center of the cell (where the heat flux is purely convective) as

$$Nu \sim u_c T_c \tag{28}$$

To estimate the velocity fluctuations in the center of the cell the only dimensional relation (ignoring thermal diffusivity and kinematic viscosity) is the following

$$u_c \simeq (g\alpha T_c \cdot H)^{1/2} \tag{29}$$

The basic assumption of the theory is that in the B-layer the characteristic velocity fluctuation are given by the balance between the buoyancy effect of the plume and the viscous effect, while temperature fluctuations are equal to the temperature fluctuations carried out by the plumes, namely

ΔT. Furthermore, the velocity fluctuations in the B-layer equal the velocity fluctuation in the central region. Thus, we obtain:

$$u_c \simeq \delta u \simeq \frac{g\alpha\, \Delta T\, \lambda^2}{\nu}\, \delta T \simeq \Delta T \tag{30}$$

From relation (8), (28), (29) and (30) it follows that $Nu \sim Ra^{2/7}$. Let us remark that, due to (30), λ can be regarded as the cutoff-scale of velocity fluctuations. One of the most important point in the Castaing *et al.* theory is the role played by the thermal plumes, which are well identified coherent structures observed in the chaotic dynamics of thermal turbulence. Equation (30) is based on the assumption that coherent plumes do exist and are observed in the B-layer and set the characteristic velocity and temperature fluctuations in Rayleigh–Bénard systems.

A major criticism on the Castaing *et al.* model concerns the validity of Eq. (30) in the mixing layer B. Indeed, as already discussed, Eq. (30) implies a balance between viscous dissipation and buoyancy force. The buoyancy force, however, should be relevant only for scales larger than L_B, the Bolgiano scale. The analysis performed in Section II, indicates that, for scales close to the thermal boundary layer λ, L_B is much greater than λ, which implies that velocity fluctuations cannot be controlled by the overall strength of the buoyancy force. This implies that the thickness of the thermal boundary layer cannot be fixed by the balancing proposed in Castaing *et al.*

In order to overcome this criticism, a rather different approach has been proposed by Shraiman and Siggia, whose theory is based on the relevant dynamical role played by the mean flow observed in Rayleigh–Bénard cells. The onset of a mean flow is due to plumes rising from the unstable boundary layer. On the other hand, the mean flow generates a viscous boundary layer, which, in turn, control the thickness of the thermal boundary layer. It is assumed, therefore, that the thermal boundary layer is contained in the viscous layer. As we shall see, the most important assumption in the theory is that all energy dissipation is constrained inside the viscous boundary layer.

The starting point of Shraiman and Siggia theory is that, within the thermal boundary layer, there exists a balancing between the mean flow advection of horizontal temperature gradient and vertical thermal dissipation, namely

$$u \frac{\partial T}{\partial x} \simeq \chi \frac{\partial^2 T}{\partial z^2} \tag{31}$$

The velocity profile is supposed to be linear inside the viscous boundary layer (τ is a mean shear which has to be determined self-consistently)

$$u \sim \frac{z}{\tau} \tag{32}$$

From (31) and (32) we can derive a relation between the thermal boundary layer thickness (λ) and the unknown parameter τ.

$$\begin{gathered} \frac{\lambda}{\tau} \cdot \frac{\delta T}{L} \simeq \chi \frac{\delta T}{\lambda^2} \\ \lambda^3 \simeq \chi L \tau \end{gathered} \tag{33}$$

The viscous layer thickness (η) can be estimated using the requirement

$$\frac{(\eta/\tau) \cdot \eta}{\nu} \simeq 1 \tag{34}$$

Using the exact relation $\langle \varepsilon \rangle = Nu \cdot Ra$ and under the assumption that the relevant part of the energy is dissipated inside the viscous boundary layer, we obtain

$$\langle \varepsilon \rangle = Nu \cdot Ra \simeq \nu \frac{1}{\tau^2} \frac{\eta}{H} \tag{35}$$

Finally from (33), (34) and (35) we obtain $Nu \sim Ra^{2/7}$.

We want to point out that the two theories are based on two quite different physical pictures of the basic mechanisms which control heat transport in Rayleigh–Bénard system. In the theory by Castaing *et al.* buoyancy effects and their balance with dissipation control the characteristic size of temperature and velocity fluctuations. According to Shraiman and Siggia, on the other hand, buoyancy is responsible to maintain the mean flow in the system which dynamically controls temperature and velocity fluctuations.

In the next section we present further experimental and numerical results which will clarify the physics of the thermal boundary layer.

We close this section remarking that the scaling properties discussed so far, are not supposed to be asymptotic, as shown by Kraichnan[11] in the early sixties. For very large Ra numbers an asymptotic regime should emerge as can be understood by the following argument. The maximal velocity which can be reached inside the cell can be estimated as:

$$U_M \equiv (\alpha g \, \Delta T \, H)^{1/2} \tag{36}$$

Since the maximal rate of energy dissipation associated with this velocity is $U_M^3/H = \varepsilon$, it follows that $Nu \sim Ra^{1/2}$. We notice that for such a scaling regime the Bolgiano scale would became Ra independent. Asymptotic prediction of 1/2 exponent can also be derived as a rigorous upper bound of heat transport (see Doering and Constantin).

From the experimental point of view, some evidences of transition to the asymptotic regime have been recently reported.[15]

IV. NUMERICAL RESULTS

An important insight in the physics of thermal convection can be obtained by direct numerical simulation (DNS) of Rayleigh–Bénard cells. An obvious limitation of DNS is that the available range in Ra is usually much smaller than in laboratory experiments. Nevertheless, DNS are extremely useful in understanding the validity of different physical assumptions and, occasionally, in checking the predictions on physical quantities which are difficult to measure in a real experiments. In this section we discuss a set of new numerical simulations aimed at understanding the correct physics of heat transport in Rayleigh–Bénard.

As already pointed out, numerical simulation are confined to rather small Ra number with respect to those available in real experiments. In ref. 1 the 2/7 scaling exponent was observed for Ra larger than 10^6. The same scaling exponent was reported by De Luca *et al.*[16] in 2D numerical simulation of Rayleigh–Bénard.

Here we show that the 2/7 scaling exponent is clearly observed, in 3D, even at rather low Ra number. We have performed a number of numerical simulations using a LBE (Lattice Boltzmann Equation)[14] code on a parallel supercomputer[17] for a 3D Rayleigh–Bénard cell of aspect ratio 1. For a detailed description of the numerical code used we refer the reader to ref. 7. For the temperature field, we have imposed adiabatic boundary conditions on vertical walls, while top and bottom walls have been kept at fixed temperature (respectively $-\Delta T/2$ and $\Delta T/2$). Velocity boundary conditions are free-slip on vertical walls and no-slip (zero velocity) on top/bottom walls. All run has been done at $Pr = 1$ and cover a range of about one order of magnitude in Ra around $Ra \sim 10^6$.

We have performed runs at different Ra in order to measure the $Nu(Ra)$ dependence. We found a clear $Nu \sim Ra^{2/7}$ scaling, see Fig. 1.

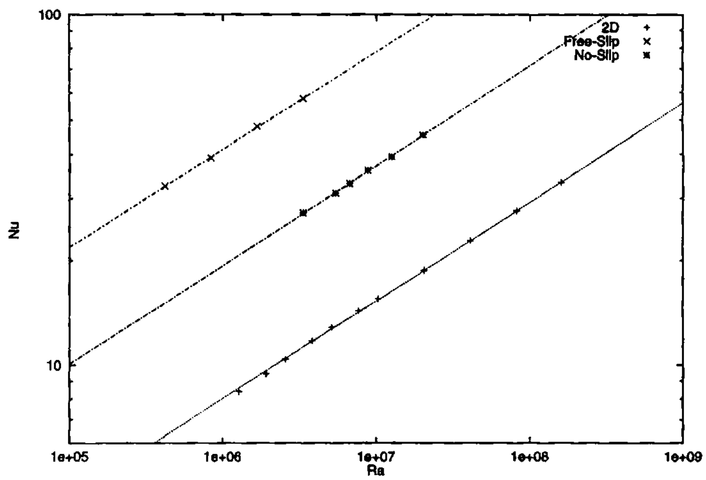

Fig. 1. Scaling of Nu versus Ra. Data + are taken from ref. 16 for a 2D numerical simulation (fitted slope 0.280 ± 0.001). Data $*$ from our numerical simulations with no-slip boundary conditions (fitted slope 0.283 ± 0.003). Same for $\times$ but with free-slip boundary conditions on top/bottom walls (fitted slope 0.277 ± 0.003).

The results shown in Fig. 1 together with those reported by De Luca *et al.*, clearly indicate that the scaling exponent does not depend on the dimensionality of the system. This implies that arguments based on three dimensional Kolmogorov theory of turbulence must be ruled out.

As discussed in the previous section, the viscous boundary layer near the top and the bottom of the Rayleigh–Bénard cell plays an important role in the model suggested by Shraiman and Siggia. In order to clarify this point, we have performed a new set of numerical simulations, with free-slip boundary conditions for the velocity on the top and bottom walls. The choice of this kind of boundary conditions allowed us to completely remove the velocity boundary layers, while keeping a dissipation of kinetic energy inside the bulk of the cell. In Fig. 1 we have also reported the scaling of Nu versus Ra for the free-slip case. A clear scaling is observed, again, with slope close to 2/7.

The numerical simulations discussed above indicate that the viscous boundary layer plays a marginal role in determining the scaling exponent of heat transport. We want to remark that in the theory proposed by Shraiman and Siggia one of the basic physical assumptions is that most of the energy dissipation occurs in the viscous boundary layer. Certainly this is not the case in the numerical simulation reported in Fig. 1 (free slip boundary condition), although the scaling exponent of Nu does not change. Our findings, therefore, seem to rule out the approach proposed by

Shraiman and Siggia, at least for what concerns the assumption on the energy dissipation.

V. BOLGIANO LENGTH AND NON-HOMOGENEOUS CONVECTIVE CELL

We have seen in Section III that one of the major criticism on the theoretical model proposed by Castaing *et al.* concerns the relevance of buoyancy force in the thermal boundary layer. The criticism is based on the fact that the Bolgiano scale L_B is much larger than λ for large Ra number. By using the observed scaling $\lambda \simeq Nu^{-1} \simeq Ra^{2/7}$, together with Eqs. (23), (24) and (20) we obtain $L_B \simeq Ra^{-3/28}$. This result show that the ratio L_B/λ increases as $Ra^{5/28}$. From a physical point of view, the Bolgiano scale represents the scale at which energy is injected in the system as "potential energy." Buoyancy force converts this energy in kinetic energy.

The analysis made in Section II was appropriate for a homogeneous/isotropic convective cell. However, Rayleigh–Bénard convective cell is not homogeneous neither isotropic. We can slightly generalize the analysis of Section II by assuming that turbulence in the Rayleigh–Bénard is "locally" homogeneous and isotropic. It follows that we must interpret the various quantities (as for example the energy dissipation $\varepsilon(z)$, the Bolgiano length $L_B(z)$ and so on) as depending locally on z: the distance from the bottom wall.

Following this approach, we can introduce a local (but plane-averaged) Bolgiano length as the following:

$$L_B(z) \equiv \frac{\varepsilon(z)^{5/4}}{N(z)^{3/4}\,(\alpha g)^{3/2}} \tag{37}$$

Using direct numerical simulations, we can obtain the behavior of $\varepsilon(z)$ and $L_B(z)$ in the Rayleigh–Bénard. In Fig. 2 we report, the energy dissipation averaged on horizontal planes as a function of z. As it is evident, energy is not evenly dissipated inside the cell. In Fig. 3 the typical behavior of the Bolgiano length, $L_B(z)$, as obtained from definition (37), is shown.

We remark that while the Bolgiano length grows inside the bulk of the cell (in particular in the center of the cell it reaches its maximum, nearly equal to the size of the cell itself) it is relatively small near the top/bottom boundaries.

This observation solves the apparent inconsistency raised in Section III where we noticed that $\lambda \ll L_B$: while the global Bolgiano length, L_B, can be much greater than the thermal boundary layer thickness, locally, the Bolgiano length, $L_B(z)$ has its minimum value around the boundary layer thickness itself.

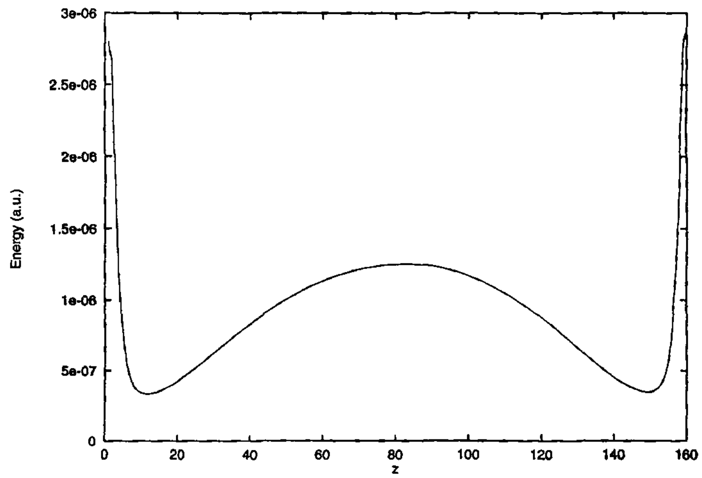

Fig. 2. Typical behavior of the energy dissipation (in arbitrary units), $\varepsilon(z)$, inside the cell.

The results shown in Figs. 2 and 3 suggest that close to the thermal boundary layer the buoyancy force is the dominant effect in the system. Thus, velocity and temperature fluctuations should be described by the Bolgiano scaling (17) and (18). On the other hand, the thermal boundary layer can be interpreted as the scale at which dissipation becomes relevant with respect to buoyancy force. By using (17) and (18) and balancing the second term (namely the forcing) in Eq. (12) with the third term, we can

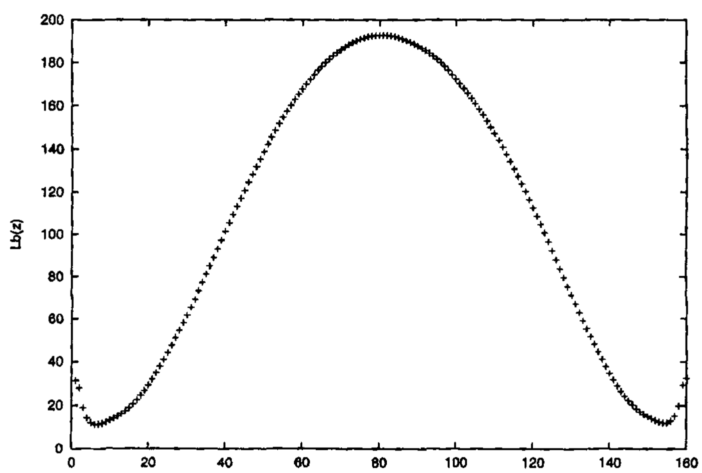

Fig. 3. Typical behavior of the Bolgiano length (in lattice units), $L_B(z)$, inside the cell.

introduce the Bolgiano dissipation length r_B. After some algebraic computation we obtain

$$r_B \simeq N^{-1/8} \tag{38}$$

r_B is equivalent to the Kolmogorov length η for turbulent flows without buoyancy force. In a Rayleigh–Bénard cell, temperature dissipation, N, is confined in the thermal boundary layer. Therefore, Eq. (38) can be used locally near the thermal boundary layer assuming that N is the global mean rate of temperature dissipation.

Looking again at the results shown in Figs. 2 and 3, we are tempted to assume that in Rayleigh–Bénard turbulence the thickness of the thermal boundary layer adjusts itself in such a way that it becomes equal to the Bolgiano dissipation length, i.e.,

$$\lambda \simeq r_B \tag{39}$$

Using Eq. (38) and (24) we immediately obtain $Nu \simeq Ra^{-2/7}$. Our ansatz (39) generalizes the approach proposed by Castaing *et al.* by using the basic Eqs. (12) and (13).

By using the definition of $L_B(z)$ and the Bolgiano scaling (17) and (18) we can compute the value of the Bolgiano length at scale λ. We estimate the rate of energy dissipation as $\varepsilon(\lambda) \simeq (\delta v(\lambda)^2)/\lambda^2$. Using (24) for the estimate of N, we finally obtain:

$$L_B(\lambda) \simeq N^{-1/4}\lambda^{-1} \tag{40}$$

From Eq. (40) we immediately see that the ansatz (39) implies:

$$L_B(\lambda) \simeq \lambda \tag{41}$$

Equation (41) should be considered a prediction of the theory so far discussed. We show in Fig. 4 the values of $L_B(\lambda)$ and λ as obtained in our numerical simulations. As one can see, the numerical results are quite consistent with the prediction (41).

Inside the thermal boundary layer, we can assume, following Shraiman and Siggia, that the velocity profile is linear in z and that the advection of the mean flow balances the thermal dissipation as expressed in Eq. 31. By using the same approach suggested by Shraiman–Siggia, we obtain that the mean shear inside the thermal boundary layer should be proportional to λ^{-3}. This implies that the mean rate of energy dissipation $\varepsilon_{\text{B.L.}}$, integrated inside the thermal boundary layer, is proportional to λ^{-5}. Therefore we finally obtain:

$$\varepsilon_{\text{B.L.}} \sim Ra^{1/7}\varepsilon_{\text{TOT}} \tag{42}$$

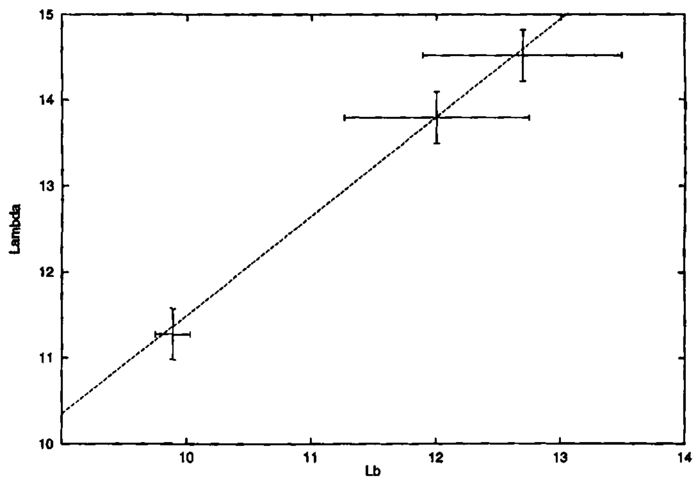

Fig. 4. Scaling of λ versus r_B. The points corresponds to Ra values of $3.34 \cdot 10^6$, $6.68 \cdot 10^6$, $1.67 \cdot 10^7$. The straight line is a linear fit, yielding $\lambda = 1.15 \cdot r_B$.

Equation (42) is consistent with the numerical simulations, see Fig. 5 where we show $\varepsilon_{\mathrm{B.L.}}$ plotted against $\varepsilon_{\mathrm{TOT}}$ in a log–log plot. Equation (42) puts bound on the range of Ra where the 2/7 scaling regime is observed. More precisely, a transition to the asymptotic scaling predicted by Kraichnan should be observed for values of Ra such that $\varepsilon_{\mathrm{B.L.}} \sim \varepsilon_{\mathrm{TOT}}$. According to

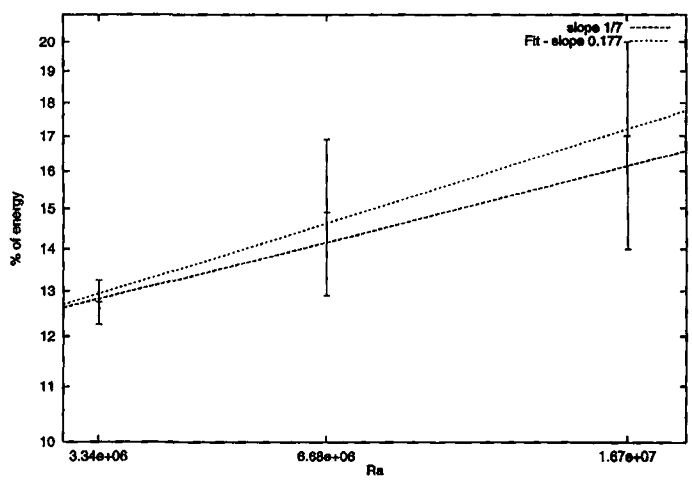

Fig. 5. Behavior of the fraction of energy dissipated inside the viscous boundary layers over the total energy dissipate inside the cell. The line are, respectively a fit with the expected slop 1/7 and with arbitrary slope (~ 0.177).

Fig. 5, the critical Ra number for this transition is predicted at $Ra \sim 10^{11 \pm 1}$ for Prandtl number of order 1. This prediction is in reasonable agreement with recent experimental results found by Chavanne *et. al.*[15]

VI. CONCLUSIONS

The most important result shown in this paper is that the observed scaling of Nu versus Ra in Rayleigh–Bénard systems can be explained by assuming that the thickness λ of the thermal boundary layer is controlled by (and also approximately equal to) the Bolgiano dissipation scale r_B, i.e. the scale at which buoyancy forces are balanced by the dissipative effects. In order to justify our assumption we have introduced a local Bolgiano scale $L_B(z)$ based on the local values of energy and temperature dissipation. By using direct numerical simulations of Rayleigh–Bénard turbulence, we have found that $L_B(z)$ is rather small near the thermal boundary layers and becomes equal to the cell size near the center of cell. This is a key point in our analysis because it allows to use Bolgiano scaling in order to compute the dissipation scale. Previous investigations started with the observation that L_B is much larger than λ and, hence, for $z \simeq \lambda$ Bolgiano scaling should be ruled out. Using the assumption that $\lambda \simeq r_B$, we predict that $L_B(\lambda) \simeq \lambda$. Such a relation has been verified numerically. Our findings support and, in some sense, generalize the model proposed few years ago by the Chicago group.[1] Let us also remark that our results agree quite well with the observed Bolgiano scaling in direct numerical simulations of Rayleigh–Bénard systems.

We have not discussed the dependence of Nu on the Prandtl number Pr. Both the model proposed by Castaing *et al.* and Shraiman–Siggia predicts a Pr dependency which scales as $Pr^{-2/7}$. Experimentally, for $Pr \leqslant 1$ the scaling of Nu versus Ra seems to be independent of Pr, while for $Pr < 1$ it seems that $Nu \simeq Pr^a$ with a small and positive. The whole behavior is however not completely clear and it is still under investigation (see ref. 18 for very recent results). At any rate, a negative scaling exponent in the Pr number is ruled out by existing experimental observations. The model we have proposed in this paper can partially explain the experimental results. Indeed, the Bolgiano dissipation scale r_B can be a rather complex function of Pr because either kinematic viscosity or thermal diffusivity can enter in the computation of r_B. Moreover, if one takes into account intermittent fluctuations and multiscaling effects on the dissipation scale, the computation of r_B may explain the observed dependence on Pr. This problem deserves more investigations in the future.

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